INTEGIRLS Fall/Winter 2024 Overview

INTEGIRLS Problem Writing Team*

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This is the Solutions Document for INTEGIRLS Fall/Winter 2024 Chapter Competitions. If you don't understand a given solution, please feel free to click on this link to send us an email with your question.

*Specific attributions can be found later in the document

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2 Solutions

2.1 HS Indiv

1. Since all of the squares, the square root and the absolute value should have value larger or equal to 0, and they add up to zero, we have

$$a-5=0, 2b-4=b, 3c-3=0 \implies a=5, b=2, c=1.$$

Therefore, the value of $\frac{a^2+b^2+c^2}{a+b+c} = \frac{25+4+1}{5+2+1} = \boxed{\frac{15}{4}}$.

Stephanie

2. The first two statements imply that both of their numbers are between 1 and 19 inclusive. The third statement implies Alice's number is 19 because they chose distinct numbers. Therefore, Bob's number is 9 and the sum of the squares of their numbers is $9^2+19^2=\boxed{442}$.

Jiseop

3. In one hour, Sam and Maria can build $\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$ of the house. This means to build the whole house, it will take $\frac{3}{2}$ hours, or one and a half hours, which is equivalent to $\boxed{90}$ minutes.

4. There are 6 letters in the word "yellow," so there are $\frac{6!}{2}$, or 360 ways to rearrange the letters. If we want to separate the "l"s, we can subtract the number of rearrangements with them together from the total number of arrangements. To find the number of arrangements where they are together, we can count them both as 1 letter. This means there are 5!, or 120 arrangements. 360 - 120 = $\boxed{240}$.

Drishti

5. The sum of the values on the 2 die can either be 6 or 12 since the greatest sum possible is 12. The possible ways to get either of these two sums are $1-5, 2-4, 3-3, 4-2, 5-1, 6-6$. This is 6 ways out of $6\cdot 6=36$, so the probability is $\frac{6}{36}=\boxed{\frac{1}{6}}$.

Ryka

6. Since the angles of a triangle add up to 180°, we know that $\angle ABC+\angle ACB=180-54=126$. We know that $\angle EBC=\frac{\angle ABC}{3}$ and $\angle GCB=\frac{\angle ACB}{3}$, so $\angle EBC+\angle GCB=\frac{\angle ABC+\angle ACB}{3}=\frac{126}{3}=42$. Since $\angle EBC$ and $\angle GCB$ are angles in triangle XBC, we have $\angle EBC+\angle GCB+\angle BXC=180$, so $\angle BXC=180-42=\boxed{138}$.

Aaron

7. Note that

$$\begin{aligned} &27^{\sqrt{y}} \\ =&(x^{\sqrt{y}})^{\sqrt{y}} \\ =&x^y=(\sqrt{x})^{2y}=81 \end{aligned}$$

Therefore, $\sqrt{y}=\log_{27}(81)=\frac{4}{3}, y=\frac{16}{9}$. Follows that $x=9\sqrt[4]{3}$. Multiplying them and we can get $xy=\boxed{16\sqrt[4]{3}}$.

Stephanie

8. We have 3 cases to consider here. Person q (our single person) has a person directly to their left and directly to their right. The people paired with each of these two people can either be sitting across the single person from their pair-mate or on the same side of the single person as their pair-mate. Each of these cases (two acrosses, two same sides, one of each), can be fairly quickly written down and reasoned through, but the only one that does not lead to a contradiction of some sort is the two acrosses case.

Specifically, let us have the following setup around person q: $abcqacb$, which ends up being the only setup that allows us to create a valid seating arrangement. For our other four pairs, the only way they can seat themselves in the eight remaining chairs is $abcdabcd$. Therefore, to create a valid seating, we must assign one of the 15 seats for person q to sit in, and then determine how to fill the remaining 7 pairs in our arrangement with 7 couples. Total, therefore, we have $15 \cdot 7! = \boxed{75600}$ arrangements of people.

Scarlet

9. Knowing that the area of a triangle is $\frac{1}{2}ab \cdot \sin C$, we find the area of ABC is $\frac{7}{2} \cdot \sin A$, meaning $\sin A$ is some integer multiple of $\frac{2}{7}$ that is less than 1. Therefore, it must be either $\frac{2}{7}, \frac{4}{7}$, or $\frac{6}{7}$. Since $\cos^2(x) = 1 - \sin^2(x)$, our maximum and minimum for $\cos^2(x)$ will occur at our minimum and maximum of $\sin(x)$. Plugging $\frac{6}{7}$ and $\frac{2}{7}$ into $1 - \sin^2(x)$, we have $\frac{13}{49} + \frac{45}{49} = \frac{62}{49}$, giving us a final answer of $62 + 49 = \boxed{111}$.

Scarlet

10. Assume that the normalability of winning an award for a single lottery card is p. Then, the following equation
$$\binom{10}{2} p^2(1-p)^8 = \binom{10}{1} p(1-p)^9$$
holds. Solving this equation, we get that
$$9p = 2(1-p) \implies \boxed{p = \frac{2}{11}}$$

Stephanie

11. It's somewhat commonly known (or you can list it out) that there are 25 primes less than 100, so her number is 98, 99, or 100. We get that the sum of factors for 98, 99, and 100, are $(1+2)(1+7+49) = 171$, $(1+3+9)(1+11) = 156$, and $(1+2+4)(1+5+25) = 217$, so her number is $\boxed{99}$.

Sophia

12. We use Ptolemy's theorem on $ABCD$. Let $x = AC$ and $y = AB$. Observe that $AB = BC$. By Ptolemy's, we have
$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$
$$\implies 7y + 5y = 9x.$$
Hence, $\frac{AC}{AB} = \frac{x}{y} = \frac{4}{3}$, and therefore our answer is $4 + 3 = \boxed{7}$.

Amogh

13. Because of power of a point, AD will equal AF, BE will equal BD, and CF will equal CE Therefore, the side lengths of ABC are 17, 19, and 12. Using Heron's formula, we have that the area of the triangle is $\sqrt{(s)(s-a)(s-b)(s-c)} = \sqrt{(24)(5)(7)(14)} = 7 \cdot 4 \cdot sqrt15 = 28\sqrt{15}$. Our answer, therefore, is $28 \cdot 15 = \boxed{420}$.

Scarlet

14. Either 2, 4, or 6 goldfish share that name for parity reasons. Then, if 2 goldfish share a name, then there are $3 \cdot 3 = 9$ ways. If 4 goldfish share a name, there are also $3 \cdot 3 = 9$ ways. If all 6 goldfish share a name, there is 1 way. Thus, $\frac{m}{n} = \frac{9}{19}$ so $m + n = \boxed{28}$.

15. Since $abc = 1$, the fractions can be simplified to $\frac{2x}{b+1+bc} + \frac{2bx}{bc+b+1} \frac{2bcx}{abc+bc+b} = \frac{2(1+b+bc)x}{bc+b+1} = 2x$. Solving, we get that $x = \frac{1}{2}$ so $p + q = \boxed{3}$.

Ada

16. Induct on the number of operations. For the base case, with $n = 0$, the only position is the origin, and with $n = 1$, there are 4 possible positions the robot can reach. Assume that for n operations, there are $(n+1)^2$ distinct positions. Now consider $n+2$ operations. The robot can land on any points reachable with n operations by making reducible steps (e.g., WSSE can be reduced to SS, effectively reducing the number of operations). Without reducible steps, the robot either moves in a single direction or in two directions (if a third direction is included, reducible steps would exist). In the single direction case, 4 points can be reached. If the direction changes in the middle, fixing the first direction, there are $n + 1$ choices for when to change direction, resulting in $4 \cdot (n + 1) = 4n + 4$. Therefore, the total number of positions is $4n + 4 + 4 + (n + 1)^2 = (n + 3)^2$. This confirms the induction, proving that with n operations, there are $\boxed{(n+1)^2}$ possible positions.

17. We consider the area in two main parts: the rectangular parts along the edges and the sectors formed by when the dog is at a vertex. For the rectangles along the edges, each rectangle has width the length of the side and height of 12, so the total area of these rectangles should be the perimeter multiplied by 12, which is $54 \cdot 12 = 648$. Now, for each sector, the angle is also the exterior angle of that vertex and the radius of the largest possible sector is 15. So adding up the sectors, we get a full circle with radius 15, thus, area 225π. Therefore, $a + b = 648 + 225 = \boxed{873}$

Sophia

18. By Vieta's, $a + b + c = -14$ so $a^2 + b^2 + c^2 + 2abc + 2bc + 2ac = 196$. Also $ab + bc + ac = 34$ so $a^2 + b^2 + c^2 - ab - bc - ac = 196 - 3 \cdot 34 = 94$. Lastly, we know $abc = -12$. Since $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) + 3abc$, we get a final answer of $2024 - 14 \cdot 94 + 3 \cdot -12 = 2024 - 1352 = \boxed{672}$.

Sophia

19. There are $4 \cdot 3 = 12$ ways to choose the suits containing 3 and 2 cards. For the suit of 3 cards, there are $\binom{13}{3} = 286$ ways; for the suit of 2 cards, there are $\binom{13}{2} = 78$ ways. Thus, there are a total of $12 \cdot 286 \cdot 78 = 267696$ ways so the sum of digits of X is $\boxed{36}$.

Sophia

20. Rearrange to get $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 5$, $z + \frac{1}{x} = 6$. Now, multiply these three equations and subtract each from the product to obtain $xyz + \frac{1}{xyz} = \boxed{105}$.

    Another, more motivated way is to write the desired expression as $xyz + (4 - x)(5 - y)(6 - z)$, after which the xyz terms cancel and the other terms are not too difficult to find.

Amogh

21. Since we're not dealing with big numbers, we can just list out our possible pairs:

    - 1 can pair with $2, 4$, or 6
    - 2 can pair with $1, 3$, or 5
    - 3 can pair with $2, 4$, or 8
    - 4 can pair with $1, 3$, or 7
    - 5 can pair with $2, 6$, or 8
    - 6 can pair with $1, 5$, or 7
    - 7 can pair with 4 or 6
    - 8 can pair with 3 or 5

Since 7 and 8 each only have two options for numbers to pair with, we can split our arrangements into 4 cases based on which numbers pair with 7 and 8; from this setting, this rest of our pairs fall into place easily. If 7 is paired with 4 and 8 is paired with 3, then 1 can pair with either 2 or 6 to still leave a valid final pair. If 7 is paired with 4 and 8 with 5, the only possible completion pairs 1 with 6. If 7 is paired with 6 and 8 is paired with 3, we are only left pairing 1 with 4. In our final case, $(8, 5)$ and $(7, 6)$ leaves us with two possible cases (pairing 1 with 2 or 4 and letting the last pair fall into place. Adding our cases together, we find $\boxed{8}$ total solutions.

Scarlet

22. Observe that $AE = AD = BC$, so $ADBC$ and $AECB$ are isosceles trapezoids. By Ptolemy's, we have $AB \cdot CE + AE \cdot BC = AC^2$. Additionally, $CD = AB$. Hence, $CD \cdot CE = AC^2 - AE \cdot BC = AC^2 - BC^2$. Therefore, the answer is $\boxed{31}$.

Alex

23. Define $Q(x) = P(x - 3)$ and $R(x) = Q(x) + Q(-x)$. Then, the cubic and linear coefficients in R are equal to 0. Additionally, $R(0) = 2$ and $R(1) = 4$ from the problem condition. This implies that $R(x) = 2x^2 + 2$, and so $P(5) = P(5) + P(1) - P(1) = Q(2) + Q(-2) - P(1) = R(2) - P(1) = 10 - 7 = \boxed{3}$.

Amogh

24. For the time being, let us ignore the 4—we can just use $1, 2,$ and 3 and then multiply at the end to account for the fact that 1s could have been 4s instead (since $4 = 3 + 1$). Let us further notice that only 4 combinations of three of these values can add to a multiple of three: $(1, 1, 1), (2, 2, 2), (3, 3, 3)$, or $(1, 2, 3)$. If we consider our central column and one of our diagonals, we can split into nine cases. Each can be–including ordering and accounting for the forced 1 in the middle–$(1, 1, 1)$, $(2, 1, 3)$, or $(3, 1, 2)$. Therefore, we have $3 \cdot 3 = 9$ cases to test. Each of these cases ends up defining a unique tri-sepcial grid, and since these cases represent all possible values of this row+diagonal combo, these nine cases are all of our grids.

1	1	1
1	1	1
1	1	1

2	1	3
2	1	3
2	1	3

3	1	2
3	1	2
3	1	2

3	2	1
2	1	3
1	3	2

1	2	3
3	1	2
2	3	1

2	2	2
1	1	1
3	3	3

2	3	1
3	1	2
1	2	3

3	3	3
1	1	1
2	2	2

1	3	2
2	1	3
3	2	1

8 of these 9 grids have 3 ones in them. 2 of these 3 ones could be changed to 4s (the center must be left a one), for a total of $2^2 = 4$ cases per grid. Similarly, in the grid of all ones, we have 2^8 ways to arrange ones and fours. Adding together, we have a total number of grids of $(8 \cdot 4) + 256 = 32 + 256 = \boxed{288}$.

Scarlet

25. Every new line placed can intersect once with each previous line. Moving right to left, each consecutive pair of intersections will create a new bounded area that is distinct from other bounded areas (either by adding to the total bounded area or splitting an old area in two). The number of consecutive pairs of intersections is equal to one subtracted from the number of previously placed lines. The third line will add one new area, the fourth will add two, and so on. This definition may seem very similar to the triangular numbers. Indeed, these sums will mean that the $n - 2$th triangular

number is the maximum number of distinct areas for n lines. The smallest triangular number above 50 is the tenth at 55. Therefore, when $n-2$ equals 10, ergo $n = \boxed{12}$, more than 50 areas can be created.

Scarlet

26. For every product ac that is not 10 mod 11, there is exactly one value of b such that $b \equiv ac \pmod{11}$. When $ac \equiv 10 \pmod{11}$, we must have $b \equiv 10 \pmod{11}$ as well, but there are no solutions in this case as no digit between 0 and 9 can be 10 mod 11.

    Thus, we just need to count the number of ordered pairs (a, c) such that ac is not 10 mod 11. Each of these ordered pairs will then correspond to a 3 digit integer.

    In total, there are 90 possibilities, because we have 9 choices for a and 10 for c. Now, for each integer c from 2 to 9 inclusive, there is exactly one a such that $ac \equiv 10 \pmod{11}$. Therefore, we must subtract 8 possibilities, giving us $90 - 8 = \boxed{82}$ as our final answer.

Emma

27. Since $a + b + c$ is even, at least one of a, b, c must be even. Without loss of generality, suppose that a is even. Then, since $|a|$ is prime, it follows that $a = 2$ or $a = -2$. We will consider these two cases separately:

    - $a = 2$. In this case, we have

    $$bc + 2b + 2c = -89 \implies (b + 2)(c + 2) = -85.$$

      We can now check all possible factor pairs. Note that we do not need to consider order due to the symmetry of $b + 2$ and $c + 2$.

      The possible pairs are: $(-5, 17), (-17, 5), (1, -85), (85, -1)$. Now, we find b and c in each of these cases.

      From $(-5, 17)$, we get $b = -7, c = 15$. From $(-17, 5)$, we get $b = -19, c = 3$. From $(1, -85)$, we get $b = -1, c = -87$. From $(85, -1)$ we get $b = 83, c = -3$.

      We see that only $(-17, 5)$ and $(85, -1)$ work. Calculating $|abc|$ in each of these cases, we have 114 and 498 as possible values.
    - $a = -2$. In this case, we have

    $$bc - 2b - 2c = -89 \implies (b - 2)(c - 2) = -85.$$

      From this, we can see that if a pair (b', c') satisfies the above equation, then $(-b', -c')$ satisfies the equation from Case 1. Conversely, if a pair (b', c') satisfies the equation from Case 1, $(-b', -c')$ satisfies the above equation. However, since $|2b'c'| = |(-2)(-b')(-c')|$, the solutions to the above equation will not yield any new values of $|abc|$.

    Therefore, the possible values of $|abc|$ are 114 and 498, so our answer is $\boxed{612}$.

Emma

28. It is well known that $ABCD$ has an incircle because $AB + CD = BC + DA$. Let M and N be the tangency points of the incircle to AB and CD. By (degenerate) Brianchon's on $AMBCND$, we conclude that AC, MN, and BD concur. Additionally, the incenter X of $ABCD$ lies on MN, so $XY \perp CD$. Therefore, we only have to compute the difference between the distances from X and Y to CD. The distance from X to CD is simply $\frac{72}{5}$, because the height of the trapezoid is $\frac{144}{5}$ (the simplest way to see this is recognizing the 29, 30, 5 triangle). The distance from Y to CD is then $\frac{144}{5} \cdot \frac{32}{59}$. Now, $\frac{144}{5}\left(\frac{32}{59} - \frac{1}{2}\right) = \frac{144}{5} \cdot \frac{5}{118} = \frac{72}{59}$, so our answer is $\boxed{131}$.

Amogh

29. We can model this question using a generating function: $(x^3 + 2x^2 + x)^{2024}$. The motivation is noting the 1-2-1 pattern in the books, and the fact that the size numbers decrease by one each time. From here, we note that the sum of the coefficients of that function is 4^{2024}.

We now want to find the sum of the coefficients of all x^k terms when $k < 6070$. Clearly, the x^{6072} term has a coefficient of 1. By Vieta's and factoring, we obtain that x^{6071} has a coefficient of 4048. We can exploit the fact that most roots are -1 to obtain that the x^{6070} term has a coefficient of $2024 \cdot 4047$. Adding this up, we note that our sum is $4^{2024} - (1 + 2024 \cdot 4049)$. We can clearly see that 4^{2024} is the nearest perfect square to our sum, so the difference between the two is $1 + 2024 \cdot 4049$, which, when taken $\pmod{1000}$, gives us $\boxed{177}$.

Alex

30. This problem was adapted from another 3-D cube problem I wrote for this contest. Note that if one vertex has value x, the adjacent vertexes all have value $2 - x$, and then the adjacent vertices to these 3 already adjacent vertices will have value x. In other words, there are "rings" of alternating equal values. Thus, we may now begin drawing the convex quadrilaterals, and we can quickly notice that the enclosed solid is just the volume of the cube subtracted by the total volumes of the 8 triangular pyramids at the corners of the cube. We want to minimize the volume we subtract, which is $\frac{2(2-x)^3+2x^3}{3} = \frac{2(8-12x+6x^2)}{3}$. The minimum to this quadratic occurs at $x = 1$, so we can simply just compute the solid when $x = 1$ to obtain $\frac{22}{3} \implies \boxed{25}$

Alex

2.2 HS Relay 1

1. Let the box be defined by the product of the intervals on the x, y, and z axes as $[0,\ 4]\times[0,\ 6]\times[0,\ 8]$ with volume $4 \times 6 \times 8$. The set of points inside the box that are not within 1 unit of any face is defined by the product of the intervals $[1,\ 3] \times [1,\ 5] \times [1,\ 7]$ with volume $2 \times 4 \times 6$. This volume is

$$\frac{2 \times 4 \times 6}{4 \times 6 \times 8} = \frac{1}{4}.$$

So the answer is 1+4 = $\boxed{5}$

2. Overall, Amber is able to lay down 3 gallons of fluid per minute. However, we also must consider that when she is 5 gallons or less from achieving her goal, she can put down 5 gallons and end, not burning any fluid. Keeping this in mind, after 10 minutes she will have 30 gallons, and after 11 minutes she will have 33 gallons. At that point, she can put down 5 gallons the next minute, finishing the pool after $\boxed{12}$ minutes.

3. Clearly the sums are between 3 and 23 inclusive. There are 8 primes $(3, 5, 7, 11, 13, 17, 19, 23)$ in this range and we want 6 of them. The sum of 1 through 12 is 78 and the sum of the 8 primes are 98, so we want to remove two that sum to 20. The possible pairs are (3,17) or (7,13). Clearly 11 and 12 make 23. So 10 and 9 must make 19. Since 8+7=15¡17, the pair (3,17) must be removed. Then 13 can be made by (5,8) or (6,7). If 13 is made by (5,8), then 11 is made by (7,4). So 7 must be made by (1,6), causing 5 to be made by (2,3). Or else is 13 is made by (6,7), 11 is made by (3,8). So 7 is made by (2,5) and 5 is made by (1,4). The first pairing gives a sum product of $132 + 90 + 40 + 28 + 6 + 6 = 302$ and the second also has sum product of $132 + 90 + 42 + 24 + 10 + 4 = 302$, so the answer is $\boxed{604}$.

4. Firstly, if the units digit is a 1, the number multiplied by 13 to achieve it is a 7. However, $47 \cdot 13 > 604$, so only $7, 17, 27,$ and 37 count for those cases here. If the ten's digit is 1, then it either between 10 and 19, 110 and 119, 210 and 219, 310 and 319, 410 and 419, or 510 and 519: giving us 13, 117, 312, and 416. Now, all numbers from 100-199 have a 1 in the hundreds place: $13 \cdot 8 = 104$ and $13 \cdot 15 = 195$. Excluding repetitions, there are a total of 4+4+8-1=$\boxed{15}$ ones. (117 gets double counted).

2.3 HS Relay 2

1. Firstly, we note that they must run 700 meters before they meet for the second time on the starting line and they stop running. Then, we note that Amogh makes 7 revolutions and Alex makes 5. On each of Amogh's revolutions, he passes Alex once, giving us the answer of $\boxed{7}$.

2. Observe that $AE = AD = BC$, so $ADBC$ and $AECB$ are isosceles trapezoids. By Ptolemy's, we have $AB \cdot CE + AE \cdot BC = AC^2$. Additionally, $CD = AB$. Hence, $CD \cdot CE = AC^2 - AE \cdot BC = AC^2 - BC^2$. Therefore, the answer is $\boxed{31}$.

3. By bashing small numbers, we see that $(2, 5)$ works as a solution. Then, we add 5 to x and subtract 3 from y to get $(7, 2)$. These are the only two cases where both x and y are positive, so the answer is $\boxed{9}$.

4. Note that $(a + 1)(b + 1)(c + 1) = 1 + a + b + c + ab + bc + ca + abc$, which gives us

$$ab + bc + ca + \frac{1}{2}(9 + 2a + 2b + 2abc) + -1(-16 - c) - \frac{39}{2} = x + \frac{x}{2} - x - \frac{39}{2} = \frac{x - 39}{2}$$

Since $(3a + 3)(2b + 2)(c + 1)$ is 6 times the expression, our answer is $3(x - 39) = \boxed{3x - 117}$.

2.4 HS Team

1. Each column in the sum contains 24 of each digit, so the sum is $24(1+2+3+4+5)(10000+1000+100+10+1) = 3999960$. Hence, our answer is $\boxed{45}$.

   Sahiti

2. Let H be the orthocenter and O the circumcenter of ABC. We claim that $H = A+B+C-2O$ holds for all triangles ABC. This follows because the centroid G satisfies $G = \frac{A+B+C}{3}$, H, G, and O are collinear, and $GH = 2HO$. Thus, $H = 3G - 2O$, which equates to $A+B+C-2O$.

   We can see that the circumcenter of ABC is just $(1,0)$ because $(-13-1)^2+48^2 = (-29-1)^2+40^2 = (51-1)^2+0^2 = 50^2$. Thus, our orthocenter lies at $(-13-29+51-1-1, 48-40+0-0-0) = (7,8)$. Our answer is $\boxed{15}$.

   Amogh

3. Because $a_5 \geq a_4+4$, we have $a_5 > a_4+3$. Similarly, we have $a_4 > a_3+3$, $a_3 > a_2+3$, and $a_2 > a_1+3$. Instead of finding a_5, a_4, a_3, a_2, and a_1, we will try to find a_5, a_4+3, a_3+3, a_2+3, and a_1+3 by determining the possible range.
   We have $25 \geq a_5 > a_4+3 > a_3+6 > a_2+9 > a_1+12 \geq 13$. Therefore a_5, a_4+3, a_3+3, a_2+3, and a_1+3 are 5 different integers between 13 and 25, inclusive. This is equivalent to selecting 5 different integers from 13 to 25 where order is not important. A selection of these 5 integers will uniquely determine a_5, a_4, a_3, a_2, and a_1. Therefore the answer is:

   $$\binom{13}{5} \bmod 1000 = \boxed{287}.$$

   Sahiti

4. Let us consider an ellipse as a transformation of a circle. For a circle with radius r, the largest triangle that fits within the circle will be an equilateral triangle with side length $r\sqrt{3}$, and therefore area $\frac{3}{4}\sqrt{3}\cdot r^2$. When the circle is stretched on one axis, this area would stretch by a factor of $\frac{a}{r}$, where a is half the length of the stretched axis. Therefore, our triangle area would be $\frac{3}{4}\sqrt{3}\cdot r^2 \cdot \frac{a}{r} = \frac{3}{4}\sqrt{3}\cdot ar$. Plugging in the numbers from the problem, we have that $15\sqrt{3}\cdot r = 45$. Simplifying, we have that $r = \frac{45}{15\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$. Our minor axis has twice this length, $2\sqrt{3}$. Therefore, our answer is $\boxed{12}$.

   Here is a proof that the largest triangle that fits within a circle is equilateral. This is well known, but the proof is included for completeness. We will show that the largest triangle within a circle of radius 1 is the equilateral triangle, after which scaling would finish the argument. If the three points are A, B, and C, and the center is O, our area is simply $\frac{1}{2}(\sin\angle AOB + \sin\angle BOC + \sin\angle COA)$. If all three angles are less than $180°$, the sine function is convex, so by Jensen's, the sum is maximized when all angles are equal at $120°$, which is precisely when ABC is equilateral. In this case, the area is $\boxed{\frac{3\sqrt{3}}{4}}$. Otherwise, if the obtuse angle is $360-2x$, our area is $\frac{1}{2}(2\sin x + \sin(360-2x)) \leq \frac{1}{2}(2\sin x) = \sin x \leq 1 < \frac{3\sqrt{3}}{4}$.

   Scarlet

5. The key idea is that the sum of the digits of n in base k is equal to $n \bmod (k-1)$. To see this, suppose the base k representation of n is $\overline{a_{d-1}a_{d-2}\dots a_0}$, where a_i is an integer from 0 to $k-1$, inclusive. Expanding using the place values gives

   $$a_{d-1}k^{d-1} + a_{d-2}k^{d-2} + \cdots + a_0.$$

   Since $k \equiv 1 \bmod k-1$, we see that this reduces to

   $$a_{d-1} + a_{d-2} + \cdots + a_0$$

   mod $k-1$, which is precisely the sum of the digits of n in base k.
   Returning to the problem, this means that $n \equiv 12 \equiv 5 \bmod 7$ and $n \equiv 9 \bmod 11$. By Chinese Remainder

Theorem, this system guarantees a unique solution for n mod 77, so everything checks out.
From the first congruence, we get that $n = 7k + 5$, where $k \geq 0$. Plugging this into the second congruence, and noting that $9 \equiv -2 \bmod 11$, we get

$$7k + 5 \equiv -2 \bmod 11 \implies 7k \equiv -7 \bmod 11,$$

meaning that $k \equiv -1 \equiv 10 \bmod 11$. We may safely divide 7 on both sides of the congruence as 7 and 11 are relatively prime.
Now, $k = 11j+10$, where $j \geq 0$. Plugging this back into our equation for n, we get $n = 7(11j+10)+5 = 77j+75$, so $n \equiv \boxed{75} \bmod 77$. Note that a solution exists 229 satisfies the requirements of the problem.

Emma

6. We first claim that for $1 \leq n \leq 48$, $\frac{n}{49-n} \equiv -1 \bmod 7$. For all n that are not multiples of 7, this is true because $n = -(49-n) \bmod 7$. For n that are multiples of 7, this reduces to $\frac{x}{7-x}$ for $x = \frac{n}{7}$, and the same argument applies.

   There are a total of $47 - 2 + 1 = 46$ terms in the given expression, so the expression is $-46 \equiv 3 \bmod 7$. Thus, if our answer is x, $3 \equiv \frac{x}{3}$, which implies that $x \equiv 3^2 \equiv \boxed{2}$.

   Amogh

7. We can write the total number of pieces as

$$\begin{aligned} a_1 + \ldots + a_{19} &= a_{10} + 9m + \ldots + a_{10} + \ldots + a_{10} + 9n \\ &= 19a_{10} + 45(m+n). \end{aligned}$$

   From the previous equation we get that $19a_{10} = 19 \cdot 107 - 45(m+n)$, so $a_{10} = 107 - \frac{45\cdot(m+n)}{19}$.

   Since 19 is prime, $m+n$ must be a multiple of 19, but $\frac{45\cdot(m+n)}{19} < 107$, so $m+n$ can only be 29 or 38. If $m+n = 19$, $a_{10} = 62$. If $m+n = 38$, $a_{10} = 17$. Therefore, the maximum possible value of a_{10} is $\boxed{62}$.

   Ada

8. We must have that $\lfloor x \rfloor \geq 7$. Suppose otherwise that $\lfloor x \rfloor \leq 6$. Then, $\lceil x \rceil \leq 7$, so $\lfloor x \rfloor \cdot \lceil x \rceil \leq 42$. We would then have that $\{x\} \geq \frac{49}{42} > 1$, which is impossible as $\{x\} < 1$ by its definition.

   To attain the minimum possible value of x, we can set $\lfloor x \rfloor = 7$. Then, $\lceil x \rceil$ is either 7 or 8. If it is 7, then $\{x\} = 0$, which doesn't work. Thus, $\lceil x \rceil = 8$.

   It follows that $\{x\} = \frac{7}{8}$, so $x = 7 + \frac{7}{8} = \frac{63}{8}$. Our answer is therefore $\boxed{71}$.

   Emma

9. We first claim that $AX = 11$. This can easily be proven by Stewart's theorem or finding the altitude from A to BC. Because $AX = AB = 11$, ABX is isosceles.

   Next, consider X', the reflection of X across AC. We claim that $ABCX'$ is cyclic. This is because $\angle ABC + \angle AX'C = \angle ABC + \angle AXC = \angle ABC + 180 - \angle AXB = \angle ABC + 180 - \angle ABC = 180$.

   We now claim that the P on AC that minimizes $PB + PX$ is $BX' \cap AC$. This is because $PB + PX = PB + PX' \geq BX'$, and equality arises when P is on BX'.

   Finally, because $ABCX'$ is cyclic, we can now use Ptolemy's on it. The value of BX' is then $\frac{11\cdot4+11\cdot12}{13} = \frac{176}{13}$, so our answer is $\boxed{189}$.

   Amogh

10. We claim that the answer is, in general, $|B-C|$. In this case, the answer is therefore $\boxed{37}$ degrees.

Let $G = EF \cap BC$ and $H = AD \cap BC$. Let X be an arbitrary point on the angle bisector of BGE.

We claim that the angle bisector of BGE is parallel to AD. Notice that $\angle ABF = \angle ADF = 90 - \angle DAC = 90 - \frac{A}{2} = 90 - \angle BAD$, so $BF \perp AD$. Similarly, $CE \perp AD$. Now, because $BF \parallel CE$, it follows that $BFEC$ is an isosceles trapezoid. Therefore, the angle bisector of BGE is perpendicular to the bases, which is parallel to AD.

Finally, $\angle CGF = 180 - \angle BGF = 180 - 2\angle BGX = 180 - 2\angle BHA = 180 - 2\left(180 - B - \frac{A}{2}\right) = A + 2B - 180 = A + 2B - A - B - C = B - C$, so we are done (note that the absolute value exists because this is a directed angle and could possibly be negative).

An alternate way to approach this problem is to recognize that EF is a rotation of BC about the circumcenter of ABC, and compute the angle of this rotation, which is equal to the answer. In particular, $\angle EDF = \angle BDC = 180 - \angle A$, which implies the rotation, and to finish, notice that the angle of the rotation is twice $\angle FDC$, which is equal to $\angle ADC - \angle ADF = B + \frac{A}{2} - 90 = \frac{B-C}{2}$. Thus, we are done.

Amogh

11. Substitute $x = a$, $y = \frac{a+b}{2}$, and $z = \frac{a+c}{2}$. Observe that a, b, and c have the same parity. Then, $3a^2 + b^2 = 8596 \cdot 4$, and $3a^2 + c^2 = 8463 \cdot 4$. Therefore, $b^2 - c^2 = 4 \cdot (8596 - 8463) = 4 \cdot 133$. This means that $(b+c, b-c) = (38, 14)$ or $(266, 2)$. The first case yields $(a, b, c) = (106, 26, 12)$ and the second yields $(a, b, c) = (74, 134, 132)$. Our tuples are therefore $(106, 66, 59)$ and $(74, 104, 103)$, and the sum of all x is $\boxed{180}$.

Amogh

12. We use complex numbers. Define a complex number that 'represents' a line to be a complex number with the same slope as the line. Let the complex numbers representing the three original lines be $1+i$, $1+2i$, and $1+5i$, and let the complex number representing the notation be $1+bi$. The complex numbers representing f', g', and h' are $1-b+i(1+b)$, $1-2b+i(2+b)$, and $1-5b+i(5+b)$. When we set the sum of the slopes of the first and third numbers is set equal to twice the middle one, we get the equation $b^3 + b^2 + b + 1 = 0$. Multiplying this by $b-1$, we obtain $b^4 = 1$, so the roots are the fourth roots of unity (other than 1). The two imaginary roots are clearly extraneous, so $b = -1$, which corresponds to lines with slopes 0, $\frac{1}{3}$, and $\frac{2}{3}$. Our answer is $2 + 3 = \boxed{5}$.

Amogh

13. We find $AM + AH$, then add $18 + 20 + 34 = 54$. We split the solution into three parts:

   (a) Find AG
   (b) Find MH
   (c) Finish

We consider the problem as an AMH-centered triangle geometry, with G the arc midpoint opposite A and O the reflection of the orthocenter of AMH about BC. Observe that $MG = GH = 34$.

We will now solve the first part. Let the midpoint of arc BC *containing* A be X. We note that $AG = OX$ because they are reflections about the perpendicular bisector of XG. Additionally, $XO^2 + OG^2 = XG^2 = 4R^2$. The circumradius of MOG is just $\frac{34 \cdot 20 \cdot 18}{\sqrt{(36)(36-18)(36-20)(36-34)}} = \frac{85}{4}$. Therefore, $XG = \frac{85}{2}$ and so $AG = OX = \sqrt{\left(\frac{85}{4}\right)^2 - 20^2} = \frac{75}{2}$. This completes the first part.

Next, we find the length MH. Observe that $\sin \frac{\widehat{MG}}{2} = \frac{17}{\frac{85}{4}} = \frac{4}{5}$. Therefore, $\cos \frac{\widehat{MG}}{2} = \frac{3}{5}$ and hence $\sin \widehat{MG} = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$. Therefore, $MH = 2R \sin \widehat{MG} = \frac{204}{5}$. This completes the second part.

Finally, we use Ptolemy's theorem on $AMGH$ to finish the problem. Observe that $AG \cdot MH = MG(AM + AH) = 34(AM + AH)$. The left-hand side evaluates to $\frac{75}{2} \cdot \frac{204}{5} = 15 \cdot 34 \cdot 3$. Hence, $AM + AH = 15 \cdot 3 = 45$. Our final answer is $45 + 72 = \boxed{117}$.

As a side note, by a pretty rough coordinate bash, it is possible to show that $AM = \frac{13}{2}$ and $AH = \frac{77}{2}$.

Amogh

14. By Cauchy-Schwarz,

$$(a^2+1)(b^2+4) \geq (ab+2)^2.$$

Additionally, for all positive real numbers x, we have that

$$x^3 \geq 300x - 2000 \leftrightarrow (x-10)^2(x+20) \geq 0.$$

Thus plugging in $x = ab+2$ and dividing both sides by $ab+2$, we get

$$(ab+2)^2 \geq 300 - \frac{2000}{ab+2}.$$

Thus equality holds in all places. Hence $ab+2 = 10 \rightarrow ab = 8$. Also, equality holds in using Cauchy-Schwarz, so $b = 2a$. Therefore, $a = 2, b = 4$, so that gives an answer of $\boxed{6}$.

Tarun

15. Let a_n be the number of tilings of a 2-by-n grid with the rightmost two cells not removed, b_n the number of tilings with only the top right not removed, and c_n the number of tilings with only the bottom right not removed. Let x_n be the total number of tilings. Then, we have

$$a_{n+1} = a_n + b_n + c_n = x_n$$
$$b_{n+1} = a_n + b_n$$
$$c_{n+1} = a_n + c_n$$
$$x_{n+1} = a_{n+1} + b_{n+1} + c_{n+1} = 3a_n + 2b_n + 2c_n = a_n + 2x_n = 2x_n + x_{n-1}.$$

Hence, the sequence x_n satisfies the recurrence $x_{n+1} = 2x_n + x_{n-1}$, and its characteristic polynomial is $x^2 - 2x - 1$. Therefore, we can write $x_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$, and solving for A and B yields

$$x_n = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2}.$$

This expression is precisely the sum of the rational terms of $(1+\sqrt{2})^{n+1}$. These terms expand as

$$\sum_{m=0}^{\lceil \frac{n}{2} \rceil} \binom{n+1}{2m} 2^m.$$

Set $n = 716$. The only terms of this sum that are nonzero mod 239 are the first term, 1, and the term $\binom{717}{478} 2^{239}$. For the latter, observe that $2^{239} \equiv 2 \bmod 239$, and

$$\binom{717}{478} \equiv \frac{717 \cdots 479}{239 \cdots 1} \equiv 3 \cdot \frac{716 \cdots 479}{238 \cdots 1} \equiv 3 \cdot \frac{238!}{238!} \equiv 3 \bmod 239.$$

Hence, the answer is $2 \cdot 3 + 1 = \boxed{7}$.

Alex, Amogh

2.5 MS Indiv

1. The answer is 7. Lana's number is $\frac{5}{3} * \frac{11}{3} = \frac{55}{9}$
   Jenny's number is $\frac{5}{3} + \frac{11}{3} = \frac{16}{3} = \frac{48}{9}$
   Substracting their numbers, we get $\frac{55}{9}$ - $\frac{48}{9}$ $= \frac{7}{9}$. Multiplying this by 9, we get $\boxed{7}$.

   Drishti

2. The answer is 76. Let the largest number be A and the smallest number be B.
   $A = \frac{5}{5+4+3} * 456 = 190$
   $B = \frac{3}{5+4+3} * 456 = 114$
   $A - B = 190 - 114 = \boxed{76}$

   Drishti

3. We know that if a number ends with 5, its square must also end with 5. For any other number, this is not true. This means, that the primes whose perfect squares end with 5 must also end with 5. The only such prime number is 5, so the answer is $\boxed{1}$.

   Drishti

4. The answer is 20 percent. For all numbers, the percentage to decrease it by is the same, so let's try this with the number 100. Increasing it by 25% will make it 125. To return it to 100, it must be be multiplied by $\frac{100}{125}$, or $\frac{4}{5}$. That means it is being reduced by $\boxed{20\%}$.

   Drishti

5. The answer is 0. Vivian runs a mile in 15 minutes, so she will only have 2 miles and 30 minutes left of the race by the time Vera starts running. However, Vera runs a mile in 10 minutes, and she will run the 3 miles in 30 minutes, finishing the race at the same time as Vivian. Therefore, they tie and there is no winner, so the answer is $\boxed{0}$ minutes.

   Jenny

6. David finished in fourth place. The two strangers in this problem have no meaning, as an order can clearly be deduced: Baker finished sixth, Annie finished fifth, Chelsea finished third, and if David finished before Annie yet after Chelsea, he must have finished $\boxed{\text{fourth}}$.

   Jenny

7. The answer is 30. There are two 6-sided dice, so there are 36 total pairs of numbers. If the first die rolls a 1, the other die can roll any number from 1-6 and be under 24. This gives 6 pairs. If the first die rolls 2 or 3, the other die can still roll any number from 1-6, giving 12 more pairs. If the first die rolls 4, the second die cannot roll a 6, because 4 * 6 = 24. The condition is that the product of the numbers must be less than 24, so this doesn't satisfy it. Hence, only 1-5 can be rolled on the 2nd die, giving 5 more pairs. If the first die rolls 5, then then 2nd die can roll 1-4 to satisfy the condition, giving 4 more pairs. If the 1st die rolls 6, then the 2nd die can roll 1-3, giving 3 pairs. Adding the total number of pairs gives us $\boxed{30}$.

   Drishti

8. The answer is 1260. Clara can arrange her clothes in $\frac{9!}{4!3!2!}$ ways since the identical shirts, pants, and shoes are indistinguishable. This is equal to $\boxed{1260}$.

   Ryka

9. After unfolding the net of the cube, notice that the opposite vertices become vertices of a 2 by 4 rectangle that are diagonal to each other. Using the Pythagorean theorem, we find that the distance between these points is

$$\sqrt{2^2 + 4^2} = \sqrt{20}$$

Squaring this, we get $\boxed{20}$.

Roselyn

10. There are a few repeated characters. The letters m, r, l, d and the space are each repeated twice, while the letter o is repeated thrice. Because of this, our answer is $3! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! = \boxed{192}$.

Amogh

11. Question 11 Solution The answer is $\boxed{72 + 72\sqrt{3}}$ cm. //

Jiseop

12. The answer is 1018072. To count the number of zeros that are at the end of 2024!, we simply compute the number of 5's in its factorization. To do this, we add the number of multiples of 5 less than or equal to 2024, the number of multiples of 25, etc. This is simply:

$$\left\lfloor \frac{2024}{5} \right\rfloor + \left\lfloor \frac{2024}{25} \right\rfloor + \left\lfloor \frac{2024}{125} \right\rfloor + \left\lfloor \frac{2024}{625} \right\rfloor = 404 + 80 + 16 + 3 = 503.$$

Raising 2024! to the power of 2024, the number of zeros would simply be multiplied by 2024, so our answer is:

$$503 \times 2024 = \boxed{1018072}.$$

Sahiti

13. The answer is 60. To minimize n, we should choose the smallest prime $x = 2$. In order for $\sqrt{n}$, $\sqrt[3]{n}$, $\sqrt[4]{n}$, and $\sqrt[5]{n}$ to all be integers, the exponent y in n must be the least common multiple of 2, 3, 4, and 5. The least common multiple of these numbers is 60, so we may conclude that $n = 2^{60}$ and $y = \boxed{60}$.

Sahiti

14. The condition is equivalent to $\overline{ABC} \equiv 4 \pmod{11}$. Therefore, the sum must be $8 \pmod{11}$ and so the minimum possible value is $\boxed{503}$. One possible construction is $125 + 378$.

Amogh

15. Notice that F is the reflection of E across the perpendicular bisector of CD. Now, $AE = DF = CF$, because F is on the perpendicular bisector of AD. Finally, this means that $AECF$ is a rectangle, which implies that $EF = AC$. The value of AC is $\sqrt{73}$ by Pythagoras, so our answer is $\boxed{73}$.

Amogh

16. Max runs 200 meters per minute faster than Leon, so she will gain 200 meters per minute on him. This means that after 10 minutes, she will be 2000 meters ahead, or exactly the length of the track. Plugging this in verifies that after $\boxed{10}$ minutes, both runners are again at the starting line.

Aaron

17. If 2x rounded to the nearest integer is 17, that means 2x must be between $\frac{33}{2}$ and $\frac{35}{2}$, so $\frac{33}{4} \leq x < \frac{35}{4}$. If 3x rounded to the nearest integer is 25, that means 3x is between $\frac{49}{2}$ and $\frac{51}{2}$, so $\frac{49}{6} \leq x < \frac{17}{2}$. Combining these restraints (taking the highest lower bound and the lowest upper bound), we find that $\frac{33}{4} \leq x < \frac{17}{2}$. Altogether, this is a range of $\frac{1}{4}$ across a total possible range of 1. Thus, Schmidt has a $\frac{1}{4}$ chance of guessing the number correctly for an answer of $1 + 4 = \boxed{5}$.

Aaron

18. By symmetry, P(5 or less blueberry muffins) = P(5 or less chocolate muffins) = P(7 or more blueberry muffins). Thus, P(5 or less blueberry muffins) + P(exactly 6 blueberry muffins) + P(7 or more blueberry muffins) = 2$\cdot$ P(5 or less blueberry muffins) + P(exactly 6 blueberry muffins) = 1. The number of ways to get 6 blueberry muffins is $\binom{12}{6} = 924$ and number of total possibilities of the muffins are

$2^{12} = 4096$. So probability of exactly 6 blueberry muffins is $\frac{231}{1024}$. Thus, the probability of 5 or less blueberry muffins if $\frac{1}{2} \cdot (1 - \frac{231}{1024}) = \frac{793}{2048}$, giving us $n - m = \boxed{1255}$.

Sophia

19.

$$10^{a-1} < 2^{2024} < 10^{a}$$
$$10^{b-1} < 5^{2024} < 10^{b}$$
$$\text{multiply the two equations, } 10^{a+b-2} < 10^{2024} < 10^{a+b}$$

Therefore, $a + b - 1 = 2024$, so $a + b = \boxed{2025}$.

Stephanie

20.

$$\begin{aligned}\sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} &= \sqrt{\frac{n!}{3!(n-3)!}} - \sqrt{\frac{n!}{2!(n-2)!}} \\ &= \sqrt{\frac{n(n-1)(n-2)}{6}} - \sqrt{\frac{n(n-1)}{2}} \\ &= \left(\sqrt{\frac{n-2}{3}} - 1\right)\sqrt{\frac{n(n-1)}{2}} = 105\end{aligned}$$

$105 = 3 \cdot 5 \cdot 7$, consider the case where $\sqrt{\frac{n-2}{3}} - 1$ and $\sqrt{\frac{n(n-1)}{2}}$ are both positive integers, 105 can be factorized as

$$1 \cdot 105, 3 \cdot 35, 5 \cdot 21, 7 \cdot 15$$

Observe that $\frac{n(n-1)}{2} > (\frac{n-2}{3})^2$, $\sqrt{\frac{n(n-1)}{2}}$ should be much larger than $\sqrt{\frac{n-2}{3}} - 1$, we first try 1 and 105 which does not work through computation.

Then we try $\sqrt{\frac{n-2}{3}} - 1 = 3$, $\frac{n-2}{3} = 16, n = 50$, $\sqrt{\frac{50 \cdot 49}{2}} = 5 \cdot 7 = 35$ satisfied the condition.

Thus when $\sqrt{\binom{n}{3}} - \sqrt{\binom{n}{2}} = 105, n = \boxed{50}$.

Stephanie

21. The discriminant of the quadratic must be a perfect square, so we require that

$$(3a^2 - 8a)^2 - 4a^2(2a^2 - 13a + 15) = a^4 + 4a^3 + 4a^2 = (a^2 + 2a)^2$$

is a perfect square. Unfortunately this is always a perfect square, so we will have to explicitly solve the quadratic. This gives

$$x = \frac{3a^2 - 8a \pm (a^2 - 2a)}{2a^2} = 2 - \frac{5}{a}, 1 - \frac{3}{a}.$$

At least one of these two are an integer. If the former is an integer, that gives $a = 1, 5$. If the latter is an integer, that gives $a = 1, 3$. Thus our values of a are $1, 3, 5$, totalling to an answer of $\boxed{9}$.

Ada

22. Lets actually work through this question backwards. Since any integer n is a factor of itself, if the sum of n's factors is less than 100, n must be less than 100. All of the primes under 99, of which there are 25, clearly work. 1 also satisfies the constraints. What about powers of primes? Squares of primes up to 7 work, giving us 4 numbers. Additionally, the cube of 3 and 3rd-5th powers of 2 work—4 more numbers. What about the product of two primes? The sum of factors for those numbers will be $(p_1 + 1)(p_2 + 1)$. 2 can be multiplied by $3, 5, 7, 11, 13, 17, 19, 23, 29$ and have the resulting factors

sum to less than 100, 3 can be multiplied by $5, 7, 11, 13, 17, 19, 23$ (2 has already been counted), 5 can be multiplied by $7, 11, 23$, and 7 can be multiplied by 11. This case gives us $9 + 7 + 3 + 1 = 20$ more numbers. For the square of a prime multiplied by another primes, we would have $({p_1}^2 + p_1 + 1)(p_2 + 1)$ as our formula. With 2 as the squared prime, we could have or single prime be $3, 5, 7$, or 11. With 3 squared, we could have 2 or 5 as our single prime. With 5 squared, we could have 2 as our single prime. Total, this case gives us 7 numbers. For a squared prime times a squared prime, the only case that works is $2^2 \cdot 3^2$, giving one more number. Finally, $3 \cdot 2^3$ and $5 \cdot 2^3$ also result in a valid numbers. Altogether, these calculations give us $25 + 1 + 4 + 4 + 20 + 7 + 1 + 2 = \boxed{64}$ numbers.

Scarlet

23. X is the reflection of H across the midpoint of BC. So, the area of BCX is the same as BCH which is half of that of ABC by the height condition. Thus, the area of ABXC is $24 + 12 = \boxed{36}$.

Sophia

24. Notice that $0 = 2^0 - 1$ and $1 = 2^1 - 1$. When $n \geq 2$, it can be seen that

$$f(n) = 3 \cdot (2^{n-1} - 1) - 2 \cdot (2^{n-2} - 1) = 2^n - 1$$

Therefore, $f(11) - f(9) + f(7) = 2048 - 1 - 512 + 1 + 128 - 1 = \boxed{1663}$.

Jiseop

25. Note that if $a_i \geq 5$, we can always replace product with $2 \cdot (a_i - 2)$ to find a greater product. Therefore, all numbers must satisfy $a_i \leq 4$, meaning that they can only be factor into factors of 2 or 3.
Notice that $6 = 2 + 2 + 2 = 3 + 3$, where $2^3 < 3^2$. Therefore, we would want to maximize the number of factors of 3. Since $50 = 3 \cdot 16 + 2$, we have maximized product $2^2 3^{16}$, where $a + b = 16 + 2 = \boxed{18}$.

Rachel

26. By the Law of Cosines, $\angle A = 60°$, so $\angle B + \angle C = 120°$. Therefore, $\angle DBC + \angle DCB = 90°$, so $\angle BDC = 90°$. Hence, D lies on the circle with diameter BC, so the distance from D to M is always $\frac{7}{2}$.
It is checkable that OM hits the arc of the circle with diameter BC inside of ABC. Additionally, $OD \geq MD - OM = \frac{7}{2} - \frac{7}{2\sqrt{3}}$. Equality arises when O, D, and M are collinear. Hence, $a = \frac{7}{2}$ and $b = \frac{49}{12}$, so our answer is $\boxed{91}$.

Amogh

27. Let $x = a^2 + b^2, y = b^2 + c^2$, and $z = c^2 + a^2$. Then

$$x^2 + y^2 + z^2 = 2(a^4 + b^4 + c^4) + 2(a^2b^2 + b^2c^2 + c^2a^2)$$

while

$$\left(\frac{x + y + z}{2}\right)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2).$$

Subtracting the equations gives

$$a^4 + b^4 + c^4 = x^2 + y^2 + z^2 - \left(\frac{x + y + z}{2}\right)^2.$$

We are given that $x = 366, y = 488, z = 46$. Then

$$x^2 + y^2 = 122^2(3^2 + 4^2) = 122^2 \cdot 5^2 = 610^2 = 372100.$$

Also,

$$x + y + z = 900,$$

so $\left(\frac{x+y+z}{2}\right)^2 = 202500$. Thus our answer is $372100 + 46^2 - 202500 = 169600 + 2116 = \boxed{171716}$.

Scarlet, Tarun

28. By plugging in 1 into the expression, we get 3, which is a prime number. However, the expression can be factored in the following way:

$$\begin{aligned} n^5 + n^4 + 1 &= n^5 + n^4 + n^3 - (n^3 - 1) \\ &= n^3(n^2 + n + 1) - (n - 1)(n^2 + n + 1) \\ &= (n^3 - n + 1)(n^2 + n + 1) \end{aligned}$$

This means that the only way for $n^5 + n^4 + 1$ to be prime is for $n^3 - n + 1$ to be prime, which only happens if $n = 0, -1, or 1$. Thus, $n = \boxed{1}$ is the only solution.

Ada

29. Note that after the first sticky note is placed, any sequence of placed sticky notes results in a distinct tower. We can also note that at any given sticky note being placed, there is one placement of that sticky note which creates a new bouncy pair. There are 3 placements that do not create a new bouncy pair. Then, to create k bouncy pairs, there are $\binom{9}{k}3^{9-k}$ ways to do so. To create an even number of bouncy pairs, we would like to find $\binom{9}{0}3^9 + \binom{9}{2}3^7 + \cdots + \binom{9}{8}3^1$. We can compute this using the clever binomial expansion $(3-1)^9 = \boxed{512}$.

Alex

30. Let E be the reflection of B across C. Then, D is the centroid of ABE and O' is the circumcenter of ABE. Therefore, X is the orthocenter of ABE. Let Y be the foot of the altitude from B to AE. Then, we must find AY. $AE = 132$, and we can compute the altitude from A to be equal to $22\sqrt{10}$. Then, the altitude from B is $24\sqrt{10}$, and the distance from A to the foot of the altitude is simply $\sqrt{77^2 - 24^2 \cdot 10} = \boxed{13}$.

Amogh

2.6 MS Relay 1

1. We can use division symbols for every number up until 1, where we use subtraction. This gets us a value extremely close to -1 that gives us a final number of $\boxed{0}$, which is minimal under an absolute value.

2. Make a table consisting of the first few values of the sequence $a_1, a_2, a_3, \ldots$:

n	1	2	3	4	5
a_n	$3^{1/4}$	$3^{2/4}$	$3^{3/4}$	$3^{5/4}32^{8/4}$	$3^{13/4}$

Note that a_n is of the general form $3^{F_n+1/4}$, where F_n denotes the n^{th} Fibonacci number. The desired value of k has the property that F_{k+1} is a multiple of 4 and is the least such value satisfying $F_{k+1} \geq 4\lfloor T \rfloor$. With $X = 21$, the least Fibonacci number that is both a multiple of 4 and which is greater than or equal to $4 \cdot 21 = 84$ is $F_{12} = 144$, hence $k = \boxed{11}$.

3. Shift the tetrahedron by $< 1, 1, 1 >$ to $(0, 0, 0)$, $(6, 0, 0)$, $(0, 0, 6)$, $(0, a - 1, 0)$. Then the volume is $\frac{1}{6} \cdot 6 \cdot 6 \cdot (a - 1) = 48$. So $a - 1 = 8$. We can see the faces are two 6-8-10 right triangles, one 6-6-6$\sqrt{2}$ isosceles right triangle, and a 10-10-6$\sqrt{2}$ isosceles triangle. Clearly, the last should be the largest. Dropping the altitude to the shortest side, we get it has length $\sqrt{100 - 18} = \sqrt{82}$. Thus, the area is $\frac{1}{2} \cdot 6\sqrt{2} \cdot \sqrt{82} = 6\sqrt{41}$ so the answer is $\boxed{47}$.

4. Since a and b are between $[1, 2]$, $(2a - b)(a - 2b) \leq 0$, giving us $a^2 + b^2 \leq \frac{5}{2} * ab$, meaning that the fraction can be at most $\frac{5}{2}$. Thus, our answer is $\boxed{7}$.

2.7 MS Relay 2

1. The probability that one die will show a number greater than 4 is $\frac{1}{3}$, while the probability that the other die will show a number that is 4 or below is $\frac{2}{3}$. Multiplying these together and then multiplying by 2 (since the larger die could go first, or the smaller one), we get $\frac{4}{9}$, so the answer is $4 + 9 = \boxed{13}$.

2. Note that $(a + 1)(b + 1)(c + 1) = 1 + a + b + c + ab + bc + ca + abc$, which gives us

$$ab + bc + ca + \frac{1}{2}(9 + 2a + 2b + 2abc) + -1(-16 - c) - \frac{39}{2} = x + \frac{x}{2} - x - \frac{39}{2} = \frac{x - 39}{2}$$

. Since $(3a + 3)(2b + 2)(c + 1)$ is 6 times the expression, our answer is $3(x - 39) = 3x - 117 = \boxed{39}$.

3. A 6-digit palindrome takes the form $\overline{abccba}$, where a, b, c are not necessarily distinct digits. We expand this using the place values, giving:

$$10^5a + 10^4b + 10^3c + 10^2c + 10b + a = (10^5 + 1)a + (10^4 + 10)b + (10^3 + 10^2)c.$$

Now, we take modulo 101. Using the fact that $10^2 \equiv -1 \pmod{101}$, we get that

$$(10^5 + 1)a + (10^4 + 10)b + (10^3 + 10^2)c \equiv 11a + 11b - 11c \equiv 0 \pmod{101}.$$

This simplifies to

$$a + b \equiv c \pmod{101}.$$

Thus, $c = 101k + (a + b)$, where k is a nonnegative integer. However, because a, b, c are all digits, $c \leq 9$, so k must be 0. The problem then reduces to counting a, b, c such that $c = a + b$.

Fix a. Since $a + b = c \leq 9$, we must have $0 \leq b \leq 9 - a$. It follows that there are $10 - a$ choices for b. Then, c is determined. To finish, sum over all possible values of a ($1 \leq a \leq 9$, since the palindrome cannot start with a 0):

$$\sum_{a=1}^{9}(10 - a) = \boxed{45}.$$

4. After drawing a diagram, we may notice that the solid looks like a cube, but "shaved" around the corners. Indeed, if we consider drawing tetrahedra on all 8 corners of the cube with 3 sides of length $\frac{1}{2}$, we see that the figure remaining is the 3-D solid we desire. Thus, our answer is $8 - \frac{2}{3} = \frac{22}{3} \implies \boxed{25}$.

2.8 MS Team

1. If the average age is 8 for 5 children, then the sum of all their ages must be 40. Since the sum of the other children's ages is 20, $2x = 20$, so $x = \boxed{10}$.

   Aaron

2. This question asks for the sum, $50 + 52 + ...68 + 70$. If we take out a factor of 2 from this, we get $2(25 + 26 + ... + 34 + 35)$. Since we know the formula for the sum of the first n consecutive integers, n(n+1)/2, we can use that to get the sum of numbers 25-35. To do this, we will find the sum of the first 24 numbers, and subtract it from the sum of the first 35 numbers.
   The sum of the first 24 numbers is $24(25)/2 = 300$.
   The sum of the first 35 numbers is $35(36)/2 = 630$.
   Substracting these, we get 330. Now, we must multiply by the factor of 2 to get the answer, $\boxed{660}$.

   Drishti

3. By the Pythagorean Theorem, we know that the length of the hypotenuse is 13. The area of the triangle is half the product of the lengths of the legs, or 30. This is equal to half the product of the base (the hypotenuse), and the height (the altitude). So, half the length of the altitude times 13 equals 30. Therefore, the lenght of the altitude is $\boxed{\dfrac{60}{13}}$.

   Ryka

4. Call the price each person was originally supposed to pay x. The total bill costs $5x$. Taking into account the person who forgot their wallet, each person pays $x + 2$, so the total bill costs $4(x + 2)$. Equating these two and solving for x, we get that $x = 8$. Plugging x back in to either expression gives us the cost of the entire meal, which is $\boxed{40}$ dollars.

   Ryka

5. The pattern for the units digit for the power of two is 2, 4, 8, 6. This pattern repeats. This means that when the exponent of two is divisible by 4, it will end with 6. 1000 is divisible by 4, so the units digit of 2^{1000} is 6. Hence, the remainder when dividing by 5 will be $\boxed{1}$.

   Drishti

6. For this problem, we will use the formula distance = speed * time. We know that the distance is 220 miles and the speed is 55 mph. Plugging this into the formula, we get that the time it will take her to drive to her uncle's house is 4 hours, or 240 minutes. Adding the 45 minute long lunch break, it will take her 4 hrs and 45 minutes. Therefore, the answer is $\boxed{285}$.

   Drishti

7. Sarah can arrange her 4 classes in $4! = 24$ ways. Megan can arrange her 3! classes in $3! = 6$ ways. The difference between these numbers of arrangements is $24 - 6 = \boxed{18}$.

   Ryka

8. The total number of choices is $6! = 720$. Consider the cases when $\overline{abc} + \overline{def}$ is odd. Then, either $\overline{abc}$ or $\overline{def}$ has to be odd. The case $\overline{abc}$ is odd has $(3!)\times(3!)=36$ choices. The same is true for the case $\overline{def}$ is odd. Therefore, the total number of choices with $\overline{abc} + \overline{def}$ even is $720 - 36\times 2 = 648$. The probability is $\frac{648}{720}$ or $\frac{9}{10}$. So the answer is $10 + 9 = \boxed{19}$

   Sahiti

9. Consider 2024 in binary, $(2024)_2 = 11111101000$. The operation $\times 2$ is equivalent to shift the digits to the left by adding 0 at the end of binary, and $+1$ changes it to 1. Therefore, the minimum number of operations should be #(digit of binary) + #(number of 1 in binary) $- 1 = 11 + 7 - 1 = \boxed{17}$.

10. Because 2 is the only even prime number, the two primes must be 2 and 89. 2*89=$\boxed{178}$.

    Janice

11. Assume that the degree measures are 1x, 2x, and 7x. Because the angles must add to 180, we get that 10x=180. Thus, x=18 and the largest degree is 18*7= $\boxed{126}$.

Janice

12. Let Alice, Bob, and Clara's numbers be a, b, and c respectively. From these statements we know:
The fourth-smallest prime is 7, which we know from listing them out.
$b + c = 7$
Similarly, one can test small numbers to find the smallest number with four divisors or recognize that it must be in the form xy where x and y are distinct primes.
$a + c = 6$
Finally, the area of a square is the side length squared.
$a + b = 9$
Adding these up, we have $2a + 2b + 2c = 22$, or $a + b + c = 11$. Subtracting $a + b$, we have $c = \boxed{2}$.

Aaron

13. Since the radius joining the center to one of the triangle's vertices lies on an angle bisector of the largest triangle, we see that $\triangle BAC$ is a $30° - 60° - 90°$ triangle. Therefore $BC = \frac{1}{2}AB = \frac{1}{2}$.

By the same reasoning, each circle C_{i+1} has half the radius of the previous one C_i. Since it is nine steps from C_1 to C_{10}, the tenth circle has radius 2^{-9}.So the answer is 1+512 = $\boxed{513}$

Sahiti

14. Given a large square of side a and a small square of side b, a rectangle can only be formed if $b = \frac{a}{2}$. Its area is

$$ab + b^2 = \left(\frac{a}{2}\right) a + \left(\frac{a}{2}\right)^2 = \left(\frac{3}{4}a^2\right).$$

From $a^2 = 16$, we have $a = 4$. The perimeter is

$$2a + 4b = 4a = \boxed{16}.$$

Sahiti

15. Since Cerise goes first, it is obvious she will win, so Lima's playing is irrelevant here. The obvious strategy here is to move 10 and 7 repeatedly until 100 cannot divide by 17 anymore. In 10 moves, Cerise will have reached 85, where she can then move 10 and 5 steps, respectively, for a total of $\boxed{12}$ moves.

Jenny